

Testing nonlocality of single-photon entanglement without a shared reference frame

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The fact that a single photon can be in a superposition of several spatial modes leads to the concept of single-photon entanglement, which has been much debated in the past. Here we discuss a simple scheme for revealing the nonlocality (hence also the entanglement) of a single-photon, which is analogous to the usual case of multiparticle entanglement. The most attractive feature of our scheme is that it does not require that the separated observers share a common reference frame. Specifically, Bell inequality violation can be obtained with certainty with unaligned devices, even if the relative frame fluctuates between each experimental run of the Bell test. These ideas are relevant from an experimental viewpoint and may significantly simplify the realization of quantum communication protocols based on single-photon entanglement.

Entanglement is now recognized as a central feature of quantum mechanics, and plays a prominent role in quantum information processing [1]. Usually considered as a property of a systems composed of multiple quantum particles, entanglement also appears in a conceptually different form that involves only a single quantum particle. This is single-particle entanglement, formally represented by a state of the form

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B) \quad (1)$$

where $|j\rangle_{A,B}$ designates a state of j particles in mode A , B respectively. Such a state can be obtained for instance by splitting a single-photon on a balanced beam-splitter, with A , B denoting the outputs of the beam-splitter. In the past, there has been much debate as to whether the state (1) is entangled or not. Although this debate seems now to be largely settled – see for instance van Enk’s discussion on the subject [2] – single-photon entanglement still attracts much attention [3, 4], in particular for applications in quantum communications [5].

Here we present a simple scheme for revealing the Bell nonlocality of single-photon entanglement, building upon recent work [6]. Since Bell inequality violation can only be achieved in quantum mechanics using entanglement, this demonstrates in the strongest possible sense that state (1) is indeed entangled. Furthermore, our scheme reveals the nonlocality (hence entanglement) of (1) essentially without introducing any additional particles. In this sense, our scheme is conceptually different from that of Ref. [2], in which single-photon entanglement is first mapped onto entanglement between two atoms, on which the Bell inequality is then finally tested. Specifically, the local measurements in our scheme are performed by combining optical displacements and single-photon detection, and a Bell inequality can be violated even with infinitesimal displacements, meaning that much less than a single additional particle is introduced on average. We show

that there is a simple analogy between our scheme and a standard optical Bell test where the entanglement is carried by several photons (e.g. entangled in polarization).

In Bell tests, it is usually necessary for the parties to communicate before the test in order to establish a shared reference frame. In our case, the local measurements are determined by two distinct quantities, namely the intensity of the optical beam used for displacements, and its phase. On the one hand, the intensity of the beam, which relates to the mean number of photons, is an absolute quantity which can be characterized locally, without any exchange of information between the remote parties. On the other hand, the phase of the beam is indeed a relative quantity which must be synchronized if the parties are to know the relation between their settings.

Here we will see that it is in fact not necessary for the distant parties to share a common reference frame for testing the nonlocality of single photon entanglement. More precisely, we will show that in our scheme, Bell inequality violation can be guaranteed even though the relative phase between the measurement settings is unknown. Moreover, it turns out that even when the relative phase between the settings fluctuates from run to run during the experiment, Bell inequality violation can be obtained with certainty, provided that the phase follows a Gaussian distribution which is not completely uniform. This robustness makes our scheme appealing both from a fundamental and applied point of view.

Conceptually, our work shows that the nonlocality of single-photon entanglement is in fact far more generic and robust than previously thought. Note that this fits in a series of recent works [7–9], exploring the robustness of quantum nonlocality without a common reference frame. The practical relevance of such schemes has been experimentally demonstrated [8, 10]. Another approach based on decoherence-free subspaces [11, 12] was also explored, although it is experimentally much more challenging [13].

Beyond the fundamental interest, our work is also rele-

vant for applications. As mentioned above, single-photon entanglement is central in quantum information protocols, e.g. quantum repeaters [14]. However, witnessing and characterising it in practice is challenging. This is partly due to the complexity of aligning a common reference frame, a central issue for experimental quantum communications which is often overlooked in theoretical works. Our results show how single-photon entanglement can be tested in a device-independent way, that is without placing assumptions on the functioning of the devices used in the protocol, and without a common frame. Hence we believe it opens interesting perspectives for applications based on single-photon entanglement.

Scheme – We consider a single photon split between N spatial modes, i.e. in the state

$$|W_N\rangle = \frac{1}{\sqrt{N}} (|0, 0, \dots, 1\rangle + \dots + |1, 0, \dots, 0\rangle). \quad (2)$$

Such states have been produced experimentally using a heralded single photon source based on spontaneous down conversion and beam-splitters [15]. Note that for two observers (2) reduces to the state (1).

For testing the nonlocality of state (2), we use the Bell inequalities introduced by Werner-Wolf-Weinfurter-Zukowski-Brukner (W^3ZB) [16]. These inequalities apply to a scenario involving N observers, each having the choice between two possible dichotomic measurements, denoted $M_0^{(j)}, M_1^{(j)}$ for party j . For this scenario, all relevant full-correlation (i.e. featuring only N -party correlation terms) Bell inequalities can be compactly expressed in a single (nonlinear) inequality

$$S = \sum_r \left| \tilde{\xi}(\mathbf{r}) \right| \leq 1, \quad (3)$$

where $\tilde{\xi}(\mathbf{r}) = 2^{-N} \sum_{\mathbf{s}} (-1)^{\mathbf{r} \cdot \mathbf{s}} \xi(\mathbf{s})$, with \mathbf{r} and \mathbf{s} being vectors in $\{0, 1\}^N$ and $\xi(\mathbf{s}) = \xi(s_1, \dots, s_N) = \langle M_{s_1}^{(1)} \dots M_{s_N}^{(N)} \rangle$ the corresponding full-correlation function. Note that in the simplest case of two parties, the above inequality is equivalent to the well known Clauser-Horne-Shimony-Holt (CHSH) inequality [17].

For the measurements, we will consider the scheme suggested in Ref. [18], namely an optical displacement followed by (non-number-resolving) single-photon detection. In Ref. [6] it was shown that (2) can violate W^3ZB using this kind of measurements for any $N \geq 2$. Assigning $+1/-1$ to the no-click and click events respectively, the measurement operator is given by $M_D = 2|\alpha\rangle\langle\alpha| - 1$, where $|\alpha\rangle$ is a coherent state. Physically, the displacements are implemented by mixing the signal with a coherent state from a local oscillator on a beam splitter with high transmission. Different measurement settings correspond to different choices for the amplitude and phase of the local oscillator, which determine the amplitude and phase of $\alpha = r e^{i\varphi}$. Restricting to the 0, 1-photon sub-

space, the operator can be put on matrix form

$$M_D(r, \phi) = \begin{pmatrix} 2e^{-r^2} - 1 & 2e^{-r^2 - i\varphi} r \\ 2e^{-r^2 + i\varphi} r & 2e^{-r^2} r^2 - 1 \end{pmatrix}. \quad (4)$$

It is instructive to compare the above operator with a standard projective qubit measurement, of the form

$$M_P(\theta, \phi) = \frac{1}{2} \begin{pmatrix} \cos \theta & e^{-i\varphi} \sin \theta \\ e^{i\varphi} \sin \theta & -\cos \theta \end{pmatrix}. \quad (5)$$

One can readily see that these two measurements are equivalent up to second order in θ , by replacing $r \rightarrow \theta/2$. In particular, displacement measurements can faithfully mimic projective qubit measurements close to the z axis of the Bloch sphere [19]. Hence our scheme can be viewed as the equivalent of the usual N -particle (N qubits) Bell test, for single-photon entanglement. We believe this further clarifies that there is no fundamental difference between single particle and multi-particle entanglement.

It is interesting to compare what is required to establish a common reference frame in our scheme with what is required in a usual Bell test based on multi-particle entanglement. In the latter, the alignment of projective qubit measurements amounts to alignment of two angles (azimuthal and polar on the Bloch sphere), which requires exchange of information between the parties. For optical displacements the situation is different, since the phase and amplitude of the local oscillator play different roles. Indeed the amplitude of the optical beam is an absolute quantity which can be characterized locally by each observer and does not require any exchange of information between the parties. On the other hand, the phase of each local beam must be aligned with respect to some shared reference phase, which provides a common clock between the parties. This can be done for instance by distributing a shared (intense) reference oscillator, although it is not trivial in experiment. As we show below, this alignment of the local oscillator can in fact be dispensed with, at the cost of slightly increasing the number of measurements performed by the parties.

Guaranteed Bell violation without a shared frame—For the sake of clarity, we start by discussing the case of two parties who employ a simple measurement strategy. The amplitudes of their measurement settings are given by 0 and r for both parties. Hence the measurement operators of party k are given by $M_0^{(k)} = M_D(0, 0)$ and $M_1^{(k)} = M_D(r, \varphi_k)$. Note that for $r = 0$ the measurements correspond to single-photon detection and the local oscillator phase is irrelevant. The correlators are then given by

$$\begin{aligned} \xi(0, 0) &= -1, \\ \xi(0, 1) &= \xi(1, 0) = -e^{-r^2} (1 - r^2), \\ \xi(1, 1) &= 1 - 2e^{-r^2} (1 + r^2) + 4e^{-2r^2} r^2 (1 + \cos(\phi)). \end{aligned} \quad (6)$$

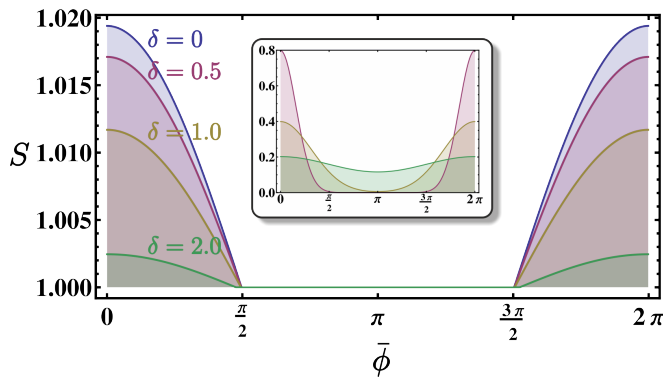


FIG. 1. W^3ZB value as a function of the center point of the distribution of the relative phase, for distributions of different width and $r = 0.1$. The top curve corresponds to a static relative phase. Recall that the local bound is 1. The inset shows the phase distributions corresponding to the three lowest curves of the main plot, taking $\bar{\phi} = 0$.

Note that the first three correlators are phase independent, and the last correlator depends only on the relative phase $\phi = \varphi_1 - \varphi_2$. Any phase acquired in transmission from the source to the parties can also be absorbed in ϕ .

If the local oscillators have not been synchronized, the local phases will be independent. Let us first consider the situation in which the relative phase ϕ is uniformly distributed on the interval $[0, 2\pi]$, but assumed to be fixed for the whole duration of the experiment, i.e. it does not fluctuate from one run of the experiment to another. From the above, and for $r \ll 1$, we have

$$S = 1 + r^2(|\cos \phi| + \cos \phi). \quad (7)$$

Hence, whenever $\phi \in]-\pi/2, \pi/2[$ we have a Bell inequality violation $S > 1$, for any value of $r > 0$ (see the blue curve in Fig. 1). We can ensure violation by introducing one additional setting for (say) the first party, measuring an observable with a shifted phase $M_1^{1'} = M_D(r, \varphi_1 + \pi)$ and then discarding either the runs where the first party employed M_1^1 or the runs with $M_1^{1'}$ (note that this is *not* postselection, since the restriction is on the settings not on the outcomes). That is, for a static, unknown relative phase, nonlocality is detected with certainty, using just 3 and 2 settings for the first and second party respectively.

Next, let us move to the case in which the relative phase fluctuates from run to run. We consider a Gaussian distribution of the phase with center $\bar{\phi}$ (which may again be unknown) and width δ . Since phases differing by an integer multiple of 2π are equivalent, the Gaussian must be wrapped onto the interval $[0, 2\pi]$. The resulting distribution is $\frac{1}{2\pi}\vartheta(\frac{\phi - \bar{\phi}}{2}; e^{-\delta^2/2})$, where ϑ is the Jacobi theta function. A static relative phase corresponds to $\delta = 0$ while a completely flat distribution is obtained for $\delta \rightarrow \infty$. Now, for a fluctuating phase the experimenters do not have access to the correlators (6) but only to their averages over the phase distribution. To compute the av-

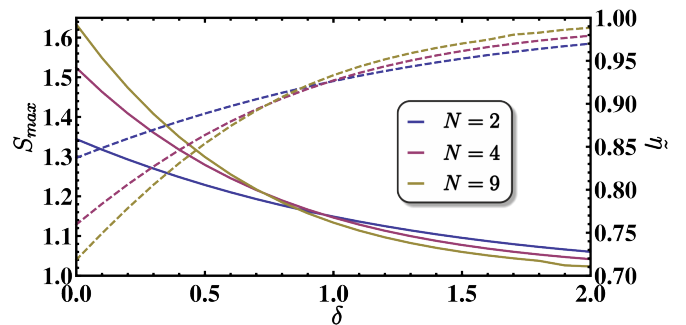


FIG. 2. Maximal W^3ZB -value S_{max} (solid) and critical efficiency η (dashed) vs. phase distribution width for $n = 2, 4$, and 9 (bottom to top for solid and top to bottom for dashed curves, for small δ).

erage of $\xi(1, 1)$, we use the fact that

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \vartheta\left(\frac{\phi - \bar{\phi}}{2}; e^{-\delta^2/2}\right) \cos \phi = e^{-\delta^2/2} \cos \bar{\phi}. \quad (8)$$

For sufficiently small r , we then find that

$$S = 1 + e^{-\delta^2/2} r^2 (|\cos \bar{\phi}| + \cos \bar{\phi}). \quad (9)$$

From this expression we see that for *any* level of fluctuations ($\delta < \infty$) the W^3ZB inequality is violated provided that $\bar{\phi} \in]-\pi/2, \pi/2[$ (see Fig. 1). As above, by adding one more measurement settings for the first party, we get Bell inequality violation with certainty, without any need for alignment, and even for arbitrary run-to-run fluctuations in the relative phase. These results demonstrate a striking robustness of the nonlocality of single photon entanglement.

Practical perspective and more parties—The robustness of the nonlocal correlations of single photon entanglement is certainly of interest beyond a purely conceptual point of view. However, from this perspectives it would be desirable to obtain higher Bell inequality violations compared to the above results, and to investigate the robustness of the violation to loss.

Higher violations are indeed possible, by allowing all measurements to have non-zero intensity (i.e. $r > 0$). It turns out that it is always optimal for all parties to use the same amplitude settings, so we do not need to allow for different settings for each party. We thus compute S for different numbers of parties, each party k using measurements $M_0^{(k)} = M_D(r_k, \varphi_k)$ and $M_1^{(k)} = M_D(r'_k, \varphi_k)$. The resulting expression depends on $N-1$ relative phases ϕ_k . Assuming these relative phases to follow Gaussian distributions as above, with identical widths δ and centers $\bar{\phi}_k$, we can find the maximal Bell inequality violation for both static and fluctuating phases. The results are summarised in Fig. 2 where the maximal violation is plotted against δ . Static phases correspond to $\delta = 0$. We note that for $N = 2$, the maximal violation is $S \approx 1.34$. If the phase distribution is not too broad, the maximal

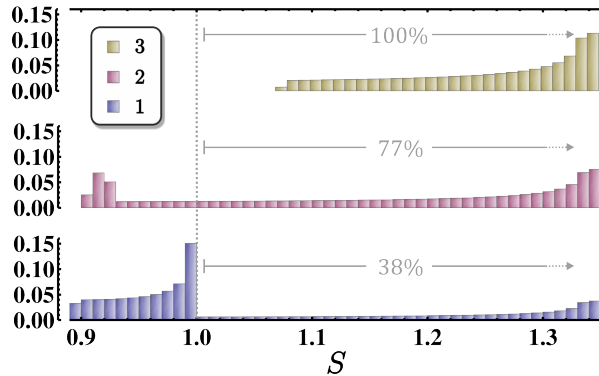


FIG. 3. Distribution of the W^3ZB -value for $\delta = 0.5$ and $\eta = 0.95$, assuming a completely unknown distribution center point, using 1, 2, and 3 local phase settings (bottom to top). The bin size is 0.01. A dotted line marks the local bound, and the probabilities of violation are also indicated.

violation increases with increasing party number, while for broad distributions it slightly deteriorates.

We can also deal with losses, e.g. in transmission or due to lossy detectors. Assuming a total efficiency η in each mode, up to local phase shifts the state after loss is $\eta|W_N\rangle\langle W_N| + (1-\eta)|vac\rangle\langle vac|$, where $|vac\rangle$ is the vacuum state. For given N and δ , we can numerically determine the threshold efficiency $\tilde{\eta}$ below which no W^3ZB violation is possible. The result is plotted in Fig. 2. As seen, for narrow phase distributions, increasing N increases the maximal violation and improves $\tilde{\eta}$. However, for broader distributions $\tilde{\eta}$ degrades slightly with N .

Both S_{max} and $\tilde{\eta}$ are only achievable exactly when the phase distribution centers $\bar{\phi}_k$ are known. However, for a given amount of phase fluctuation δ , by introducing sufficiently many local settings with different local phase shifts, it is always possible to violate for any $\eta > \tilde{\eta}$ even when the $\bar{\phi}_k$ are unknown. Similarly, when there is no loss, it is always possible approach S_{max} arbitrarily close. Thus Fig. 2 really does reflect achievable values without any alignment of reference frames. In Fig. 3, for two parties, we plot the distribution of S for different numbers of local phase settings for the first party, assuming that the distribution center $\bar{\phi}$ is completely unknown, i.e. uniform on $[0, 2\pi]$. The parties use the amplitude settings which maximise the violation for $\bar{\phi} = 0$. The local phases are evenly distributed in $[0, 2\pi]$, e.g. for three phase settings, the first party has six observables, one pair M_0 , M_1 with phase φ_1 , another with $\varphi_1 + 2\pi/3$, and a third with $\varphi_1 + 4\pi/3$. One sees that larger violations are more likely than smaller ones, and that violation is guaranteed by using three local phase settings.

Conclusions.—We discussed a simple scheme for testing the nonlocality of single-photon entanglement. Our scheme has the attractive feature that Bell inequality violation can be obtained with certainty even though the parties share no common reference frame. Recently, a

scheme [8, 9] with the same feature was discussed for two-particle entanglement (e.g. two photons entangled in polarization). Comparing to this scheme highlights the robustness of the present setup. Indeed, contrary to the former, the latter can provide Bell violations even in the case in which the relative frame varies from one run to another of the experiment (We checked numerically that the scheme of Refs. [8, 9] is not robust against arbitrary fluctuations of the relative frame. In particular, for $\delta > 0.68$ no violation is possible).

From a practical perspective, the robustness of our scheme should make it relevant for experimental implementations. In principle, Bell violations could be obtained here even without stabilizing the phase of the local oscillator. We believe that these ideas may simplify the experimental demonstration of one-photon entanglement and nonlocality, as well as quantum information theoretic tasks based on such states.

Finally, we note that the ideas presented here for displacement based measurements also apply to Bell tests and other quantum communication protocols based on homodyne measurements. We performed numerical studies showing that the schemes of Refs [4, 20] can also lead to Bell violations with high probability without a common reference frame, and we hope the present results will motivate further research along these lines.

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- [1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
 - [2] S. J. van Enk, Phys. Rev. A **72**, 064306 (2005).
 - [3] S. A. Babichev, J. Appel, and A. I. Lvovsky, Phys. Rev. Lett. **92**, 193601 (2004); B. Hessmo, P. Usachev, H. Heydari, and G. Björk, Phys. Rev. Lett. **92**, 180401 (2004); S. Ashhab, K. Maruyama, i. c. v. Brukner, and F. Nori, Phys. Rev. A **80**, 062106 (2009); S. J. Jones and H. M. Wiseman, Phys. Rev. A **84**, 012110 (2011).
 - [4] O. Morin, J.-D. Bancal, M. Ho, P. Sekatski, V. D'Auria, N. Gisin, J. Laurat, and N. Sangouard, arXiv, 1206.5734 (2012).
 - [5] N. Sangouard and H. Zbinden, (2012), 1202.0493.
 - [6] J. B. Brask and R. Chaves, Phys. Rev. A **86**, 010103 (2012).
 - [7] Y.-C. Liang, N. Harrigan, S. D. Bartlett, and T. Rudolph, Phys. Rev. Lett. **104**, 050401 (2010).
 - [8] P. Shadbolt, T. Vértesi, Y.-C. Liang, C. Branciard, N. Brunner, and J. L. O'Brien, Sci. Rep. **2**, 470 (2012).
 - [9] J. J. Wallman and S. D. Bartlett, Phys. Rev. A **85**, 024101 (2012).
 - [10] M. S. Palsson, J. J. Wallman, A. J. Bennet, and G. J. Pryde, Phys. Rev. A **86**, 032322 (2012).

- [11] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Phys. Rev. Lett. **91**, 027901 (2003).
- [12] A. Cabello, Phys. Rev. Lett. **91**, 230403 (2003).
- [13] V. D'Ambrosio, E. Nagali, S. P. Walborn, L. Aolita, S. Slussarenko, L. Marrucci, and F. Sciarrino, (2012), 1203.6417.
- [14] N. Sangouard, C. Simon, H. de Riedmatten, and N. Gisin, Rev. Mod. Phys. **83**, 33 (2011).
- [15] K. S. Choi, A. Goban, S. B. Papp, S. J. van Enk, and H. J. Kimble, Nature **468**, 412 (2010).
- [16] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001); H. Weinfurter and M. Zukowski, Phys. Rev. A **64**, 010102 (2001); M. Zukowski and C. Brukner, Phys. Rev. Lett. **88**, 210401 (2002).
- [17] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
- [18] K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. **82**, 2009 (1999).
- [19] For the case $N = 2$, measurements close to the z axis are very favorable for a loophole-free Bell test using 2-qubit entangled states [21]. Unfortunately, the scheme of Ref. [21] relies on identification and assignment of loss events, which is not possible with displacement measurements, since there is no way to distinguish legitimate and erroneous outcomes of vacuum detection. This explains why, in spite of the similarities, the low efficiencies achievable for two particle entanglement cannot be recovered here (see Ref. [6]).
- [20] D. Cavalcanti, N. Brunner, P. Skrzypczyk, A. Salles, and V. Scarani, Phys. Rev. A **84**, 022105 (2011).
- [21] P. H. Eberhard, Phys. Rev. A **47**, R747 (1993).